A Constructive Subdivision of TQ-space

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Let us introduce a discrete dynamical system:

$$\mathbf{s}_{k+1} = \mathbf{f}\left(\mathbf{s}_k,\, \mathbf{p}\right), \quad \boldsymbol{\phi}_{\mathbf{p}} : \mathrm{S} \times \mathrm{K} \to \mathrm{S}, \quad \boldsymbol{\phi}_{\mathbf{p}}(\mathbf{s},\, k) \equiv \mathbf{f}^k\left(\mathbf{s},\, \mathbf{p}\right), \tag{1}$$

with the properties: $\mathbf{s} \in \mathrm{S} \subset \mathbb{R}^N$, $\mathbf{p} \in \mathrm{P} \subset \mathbb{R}^L$, $\mathbf{f} \in \mathrm{C}^0(\mathrm{S} \times \mathrm{P})$, $k \in \mathrm{K} \subseteq \mathbb{Z}$, $n \in \overline{1,\, N}$, $l \in \overline{1,\, L}$, where $\mathbf{s}$ is the variable of map state, $\mathbf{p}$ is the vector of map parameters, k is discrete time. Let us also relate to the system (1) the trajectory of its state evolution $\mathbf{s}$: $\left\{\mathbf{f}^k(\mathbf{s},\, \mathbf{p})\right\}_{k \in \mathrm{K}}$, where $\mathbf{f}^k$ is the composition of functions:

$$\mathbf{f}^w(\mathbf{s},\, \mathbf{p}) = \mathbf{f}(\mathbf{f}(\ldots \mathbf{f}(\mathbf{s},\, \mathbf{p}),\, \mathbf{p}),\, \mathbf{p}), \quad \mathbf{f}^0(\mathbf{s},\, \mathbf{p}) = \mathbf{s}.$$

Let us define the main map that encodes the shape of n-th component of the sequence $\{\mathbf{s}_k\}$ in the space $\mathrm{S} \times \mathrm{K}$ in terms of the finite T-alphabet:

$$\left\{\mathbf{s}_{k-1}^{(n)},\, \mathbf{s}_k^{(n)},\, \mathbf{s}_{k+1}^{(n)}\right\} \Rightarrow T_k^{\alpha\varphi}|_n, \quad T_k^{\alpha\varphi} = \left[T_k^{\alpha\varphi}|_1,\, \ldots,\, T_k^{\alpha\varphi}|_N\right]. \tag{2}$$

For the relations that determine the map (2) see [1]. In the T-alphabet, all symbols are unambiguously divided into three separate classes $\mathrm{T}_o^{\alpha\varphi} = \mathrm{T}_0^{\alpha\varphi} \cup \mathrm{T}_s^{\alpha\varphi} \cup \mathrm{T}_c^{\alpha\varphi}$:

$$\mathrm{T}_0^{\alpha\varphi} = \{\mathtt{T0}\}, \tag{3a}$$

$$\mathrm{T}_s^{\alpha\varphi} = \{\mathtt{T1},\, \mathtt{T2},\, \mathtt{T4N},\, \mathtt{T4P},\, \mathtt{T6},\, \mathtt{T7},\, \mathtt{T8N},\, \mathtt{T8P}\}, \tag{3b}$$

$$\mathrm{T}_c^{\alpha\varphi} = \{\mathtt{T3N},\, \mathtt{T3P},\, \mathtt{T5N},\, \mathtt{T5P},\, \mathtt{T6S},\, \mathtt{T6L},\, \mathtt{T7S},\, \mathtt{T7L}\}. \tag{3c}$$

In addition to the symbols $T_k^{\alpha\varphi}|_n$ let us introduce symbols $Q_k^{\alpha\varphi}|_n$:

$$Q_k^{\alpha\varphi}|_n \equiv T_k^{\alpha\varphi}|_n \to T_{k+1}^{\alpha\varphi}|_n, \quad Q_k^{\alpha\varphi} = \left[Q_k^{\alpha\varphi}|_1,\, \ldots,\, Q_k^{\alpha\varphi}|_n,\, \ldots,\, Q_k^{\alpha\varphi}|_N\right].$$

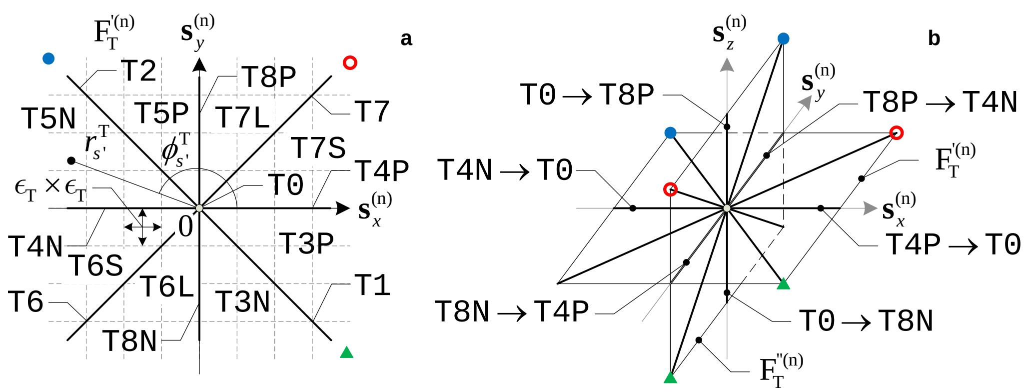
All admissible transitions comprise a set of the alphabet symbols $\mathrm{Q}_o^{\alpha\varphi} \ni Q_k^{\alpha\varphi}|_n$.

Let us introduce a representative point $P_{\mathrm{Q}\mathbf{s}}^{(n)} = \left(\mathbf{s}_x^{(n)},\, \mathbf{s}_y^{(n)},\, \mathbf{s}_z^{(n)}\right) \in \mathrm{P}_{\mathrm{TQ}}^{(n)} \subseteq \mathbb{R}^3$:

$$\mathbf{s}_x = \mathbf{s} - \mathbf{f}(\mathbf{s},\, \mathbf{p}), \quad \mathbf{s}_y = \mathbf{f}^2(\mathbf{s},\, \mathbf{p}) - \mathbf{f}(\mathbf{s},\, \mathbf{p}), \quad \mathbf{s}_z = \mathbf{f}^3(\mathbf{s},\, \mathbf{p}) - \mathbf{f}^2(\mathbf{s},\, \mathbf{p}), \tag{4}$$

which has the Cartesian coordinates that correspond to the state $\mathbf{s}$ and unambiguously denote the state as a symbol of the $\mathrm{Q}_o^{\alpha\varphi}$ alphabet (by the n-th component of the state space). Let us define a polar coordinate system $\left(\mathbf{s}_x^{(n)},\, \mathbf{s}_y^{(n)}\right) \Leftrightarrow \left(\phi_{s'}^{\mathrm{T}},\, r_{s'}^{\mathrm{T}}\right)$. Then (2) can be expressed as (see Fig. a):

$$\begin{array}{llllll}
\texttt{T0} & r_{s'}^{\mathrm{T}} = 0; & \texttt{T4N} & \phi_{s'}^{\mathrm{T}} = 4\,\phi', & \texttt{T6L} & \phi_{s'}^{\mathrm{T}} \in (5,\,6)\,\phi', \\
r_{s'}^{\mathrm{T}} > 0, & \phi' = \dfrac{\pi}{4}: & \texttt{T4P} & \phi_{s'}^{\mathrm{T}} = 0\,\phi', & \texttt{T7S} & \phi_{s'}^{\mathrm{T}} \in (0,\,1)\,\phi', \\
\texttt{T1} & \phi_{s'}^{\mathrm{T}} = 7\,\phi', & \texttt{T5N} & \phi_{s'}^{\mathrm{T}} \in (3,\,4)\,\phi', & \texttt{T7} & \phi_{s'}^{\mathrm{T}} = 1\,\phi', \\
\texttt{T2} & \phi_{s'}^{\mathrm{T}} = 3\,\phi', & \texttt{T5P} & \phi_{s'}^{\mathrm{T}} \in (2,\,3)\,\phi', & \texttt{T7L} & \phi_{s'}^{\mathrm{T}} \in (1,\,2)\,\phi', \\
\texttt{T3N} & \phi_{s'}^{\mathrm{T}} \in (6,\,7)\,\phi', & \texttt{T6S} & \phi_{s'}^{\mathrm{T}} \in (4,\,5)\,\phi', & \texttt{T8N} & \phi_{s'}^{\mathrm{T}} = 6\,\phi', \\
\texttt{T3P} & \phi_{s'}^{\mathrm{T}} \in (7,\,8)\,\phi', & \texttt{T6} & \phi_{s'}^{\mathrm{T}} = 5\,\phi', & \texttt{T8P} & \phi_{s'}^{\mathrm{T}} = 2\,\phi'.
\end{array} \tag{5}$$



From formula (5) implies that subdivision of the subspace $\mathrm{P}_{\mathrm{TQ}}^{(n)}$ into disjoint cubes:

$$\begin{array}{l}
\mathrm{S}_x^{(n)}|m_x \times \mathrm{S}_y^{(n)}|m_y \times \mathrm{S}_z^{(n)}|m_z, \\
m_\circ \in \mathbb{Z}, \quad \epsilon_{\mathrm{T}} > 0,
\end{array}
\quad \mathrm{S}_\circ^{(n)}|m_\circ = \epsilon_{\mathrm{T}} \begin{cases} \left(m_\circ - 1,\, m_\circ\right] & m_\circ > 0, \\ 0 & m_\circ = 0, \\ \left[m_\circ,\, m_\circ + 1\right) & m_\circ < 0. \end{cases}, \tag{6}$$

determines natural extension over the T- and Q-alphabets, with codes of each alphabet symbol extended with the indices $m_\circ$, that set the number (and effectively the location) of the cell. Note that this subdivision preserves the Hausdorff dimension of the symbols in the T- and Q-alphabets in the subspace $\mathrm{P}_{\mathrm{TQ}}^{(n)}$.

Let us introduce a definition.

Definition 1. The graph $\Gamma^{\mathrm{TQ}} = \left\langle \mathrm{V}^\Gamma,\, \mathrm{E}^\Gamma \right\rangle$, implies that:

$$\begin{array}{ll}
\mathrm{V}^\Gamma \subseteq \mathrm{T}_o^{\alpha\varphi}|N, \quad \mathrm{E}^\Gamma \subseteq \mathrm{Q}_o^{\alpha\varphi}|N, & \\
\forall\, T^{\alpha\varphi} \in \mathrm{V}^\Gamma \Rightarrow \exists\, \mathbf{s}_0 \in \mathrm{S} : T_1^{\alpha\varphi} = T^{\alpha\varphi}, & \forall\, \mathbf{s}_0 \in \mathrm{S} \Rightarrow \exists\, T^{\alpha\varphi} \in \mathrm{V}^\Gamma : T_1^{\alpha\varphi} = T^{\alpha\varphi}, \\
\forall\, Q^{\alpha\varphi} \in \mathrm{E}^\Gamma \Rightarrow \exists\, \mathbf{s}_0 \in \mathrm{S} : Q_1^{\alpha\varphi} = Q^{\alpha\varphi}, & \forall\, \mathbf{s}_0 \in \mathrm{S} \Rightarrow \exists\, Q^{\alpha\varphi} \in \mathrm{E}^\Gamma : Q_1^{\alpha\varphi} = Q^{\alpha\varphi},
\end{array} \tag{7}$$

is a complete symbolic TQ-image of the dynamical system $\left(\mathrm{S},\, \mathrm{K},\, \boldsymbol{\phi}_{\mathbf{p}}\right)$.

Let us also define the full TQ-space:

$$\mathrm{P}_{\mathrm{TQ}} = \mathrm{P}_{\mathrm{TQ}}^{(1)} \times \cdots \times \mathrm{P}_{\mathrm{TQ}}^{(n)} \times \cdots \times \mathrm{P}_{\mathrm{TQ}}^{(N)}, \quad \mathrm{P}_{\mathrm{TQ}}^{(n)} = \Pr_{n} \mathrm{P}_{\mathrm{TQ}}, \quad \dim \mathrm{P}_{\mathrm{TQ}} = 3\,N. \tag{8}$$

Then the consecutive subdivision of the space P_{TQ} through refinement of the cells $\mathrm{S}_x^{(n)}|m_x \times \mathrm{S}_y^{(n)}|m_y \times \mathrm{S}_z^{(n)}|m_z$ due to the parameter $\epsilon_{\mathrm{T}} \to 0$ results in a sequence of the graphs $\left\{\Gamma_{\epsilon_{\mathrm{T}}}^{\mathrm{TQ}}\right\}_{\epsilon_{\mathrm{T}} \in \mathcal{E}}$.

When certain requirements are observed with regard to the nature of ϵ_{T} change and the encoding scheme for $m_\circ$ indices, the subdivision of the space P_{TQ} can be constructively applied to studying various properties of dynamical systems.

One of the simplest subdivision schemes is the dichotomous scheme. Let us define:

$$\epsilon_{\mathrm{T}\,0}^{(n)} = \max \left| \left(\left\{ \mathbf{s}_x^{(n)},\, \mathbf{s}_y^{(n)},\, \mathbf{s}_z^{(n)} \right\} \,:\, \mathbf{s} \in \mathrm{S}_{\mathrm{a}} \right) \right|,\ \epsilon_{\mathrm{T}\,r+1}^{(n)} = \epsilon_{\mathrm{T}\,r}^{(n)}/2,\ m_\circ = \overline{-d,\, 0,\, d},\ d = 2^r, \tag{9}$$

where S_{a} is the analyzed range of definition of the phase variable $\mathbf{s}$, for example, the attractor of the system $\left(\mathrm{S},\, \mathrm{K},\, \boldsymbol{\phi}_{\mathbf{p}}\right)$. When the dichotomous scheme is used, in terms of calculation the most efficient scheme of indexing the resulting P_{TQ}-space cells is octree, which allows to ignore the "empty" areas of the space.

In conclusion, it is worth mentioning that the proposed subdivision of the space P_{TQ} provides an opportunity to significantly increase the depth of performing certain tasks in terms of the symbolic CTQ-analysis: studying the T-synchronization of chaotic systems [2], estimating TQ-complexity of the trajectories of dynamical systems [1], analyzing the TQ-bifurcations [3] in discrete systems, etc. To this end, a set of theorems and propositions has been formulated and proved.

References

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