The TQ-bifurcation in Discrete Dynamical Systems. General Properties.

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Outline section

Why is Bifurcations?

Many deterministic dynamical systems belong to a wide class of chaotic systems, in which, in turn, a fundamental phenomenon of bifurcation rearrangements of the structure of the attractor can be observed.

H. Poincare
(1854–1912)

The concept of *bifurcations* in dynamical systems was first introduced by H. Poincare, who devised this concept to describe the "splitting" of equilibrium solutions for ordinary differential equations.

Today's interpretation of the theory deals with the rearrangements of the whole system and of its invariant set of attractors, as well as the bifurcations at equilibrium points. This formulation of the "research program" can be traced back to the works of A.A. Andronov.

A. Andronov
(1901–1952)

Inception / destruction of structures, changes of synchronization modes, suppression of chaos through control, destruction of chaos / transition to chaos are accompanied by the change of symmetry and order parameters of dynamic system trajectories (in terms of modern statistical physics). But these processes are based on bifurcation mechanisms of various kinds that are realized in dynamical systems.

V.I. Arnol'd, V.S. Afraimovich, Yu.S. Ilyashenko, and L.P. Shilnikov. *Bifurcation theory, Dynamical systems*. Vol. 5. Moscow: VINITI, 1986.

Outline section

Definition of the T-alphabet

Denote the discrete dynamical system:

$$\mathbf{s}_{k+1} = \mathbf{f}\left(\mathbf{s}_k,\, \mathbf{p}\right), \quad \boldsymbol{\phi}_{\mathbf{p}} : \mathrm{S} \times \mathrm{K} \to \mathrm{S}, \quad \boldsymbol{\phi}_{\mathbf{p}}(\mathbf{s},\, k) \equiv \mathbf{f}^{k}\left(\mathbf{s},\, \mathbf{p}\right), \quad \left\{\mathbf{s}_k\right\}_{k\in\mathrm{K}},$$

$$\mathbf{s} \in \mathrm{S} \subset \mathbb{R}^N, \quad k \in \mathrm{K} \subseteq \mathbb{Z}, \quad \mathbf{p} \in \mathrm{P} \subset \mathbb{R}^L, \quad \mathbf{f} \in \mathrm{C}^0(\mathrm{S} \times \mathrm{P}), \quad n = \overline{1,\, N},\, l = \overline{1,\, L}.$$

We introduce the main map:

$$\left\{\mathbf{s}_{k-1}^{(n)},\, \mathbf{s}_{k}^{(n)},\, \mathbf{s}_{k+1}^{(n)}\right\} \Rightarrow T_k^{\alpha\varphi}|_n, \quad T_k^{\alpha\varphi} = \left[T_k^{\alpha\varphi}|_1 \ldots T_k^{\alpha\varphi}|_N\right], \quad \left\{T_k^{\alpha\varphi}\right\}_{k\in\mathrm{K}},$$

where $T^{\alpha\varphi}$ – symbol of T-alphabet:

$$T^{\alpha\varphi} \in \mathrm{T}_o^{\alpha\varphi} = \mathrm{T}_0^{\alpha\varphi} \cup \mathrm{T}_s^{\alpha\varphi} \cup \mathrm{T}_c^{\alpha\varphi}.$$

It is unambiguously divided into three separate classes:

$$\mathrm{T}_0^{\alpha\varphi} = \{\mathtt{T0}\},$$

$$\mathrm{T}_s^{\alpha\varphi} = \{\mathtt{T1},\, \mathtt{T2},\, \mathtt{T4N},\, \mathtt{T4P},\, \mathtt{T6},\, \mathtt{T7},\, \mathtt{T8N},\, \mathtt{T8P}\},$$

$$\mathrm{T}_c^{\alpha\varphi} = \{\mathtt{T3N},\, \mathtt{T3P},\, \mathtt{T5N},\, \mathtt{T5P},\, \mathtt{T6S},\, \mathtt{T6L},\, \mathtt{T7S},\, \mathtt{T7L}\}.$$

Definition of the T-alphabet

Geometry of the T-alphabet symbols:

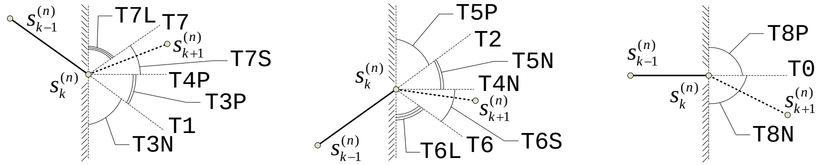

Table of the admissible transitions $T_k^{\alpha\varphi}|_n \to T_{k+1}^{\alpha\varphi}|_n \equiv Q_k^{\alpha\varphi}|_n$:

k \ k+1	T0	T1	T2	T3N	T3P	T4N	T4P	T5N	T5P	T6S	T6	T6L	T7S	T7	T7L	T8N	T8P
T0																	
T1																	
T2																	
T3N																	
T3P																	
T4N																	
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T5N																	
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T8N																	
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The TQ-Image of discrete dynamical system

Definition 1

The directed graph $\Gamma^{\mathrm{TQ}} = \left\langle \mathrm{V}^{\Gamma},\, \mathrm{E}^{\Gamma} \right\rangle$, implies that:

$$\mathrm{V}^{\Gamma} \subseteq \mathrm{T}_{o}^{\alpha\varphi}|N, \quad \mathrm{E}^{\Gamma} \subseteq \mathrm{Q}_{o}^{\alpha\varphi}|N,$$

$$\forall\, T^{\alpha\varphi} \in \mathrm{V}^{\Gamma} \Rightarrow \exists\, \mathbf{s}_0 \in \mathrm{S} : T_1^{\alpha\varphi} = T^{\alpha\varphi},\, \forall\, \mathbf{s}_0 \in \mathrm{S} \Rightarrow \exists\, T^{\alpha\varphi} \in \mathrm{V}^{\Gamma} : T_1^{\alpha\varphi} = T^{\alpha\varphi},$$

$$\forall\, Q^{\alpha\varphi} \in \mathrm{E}^{\Gamma} \Rightarrow \exists\, \mathbf{s}_0 \in \mathrm{S} : Q_1^{\alpha\varphi} = Q^{\alpha\varphi}, \forall\, \mathbf{s}_0 \in \mathrm{S} \Rightarrow \exists\, Q^{\alpha\varphi} \in \mathrm{E}^{\Gamma} : Q_1^{\alpha\varphi} = Q^{\alpha\varphi},$$

is a *Complete symbolic TQ-image* of the dynamical system $\left(\mathrm{S},\, \mathrm{K},\, \boldsymbol{\phi}_{\mathbf{p}}\right)$.

Main Article and Talk (in English)

- A.V. Makarenko, "Analysis of the time structure of synchronization in multidimensional chaotic systems", *Journal of Experimental and Theoretical Physics*, vol. 120, no. 5, pp. 912–921, 2015.
- A.V. Makarenko, "Generalized synchronization of multidimensional chaotic systems in terms of symbolic CTQ-analysis", *The Proceedings of 8th Chaotic Modeling and Simulation International Conference*, Paris, ISAST, IHP, 2015, pp. 477–490.
- A.V. Makarenko, "Estimation of the TQ-complexity of chaotic sequences", *IFAC-PapersOnLine*, vol. 48, no. 11, pp. 1049–1055, 2015.
- A.V. Makarenko, "Analysis of phase synchronization of chaotic oscillations in terms of symbolic CTQ-analysis", *Technical Physics*, vol. 61, no. 2, pp. 265–273, 2016.

Outline section

The definition of the TQ-bifurcations

Definition 2

TQ-bifurcation in the discrete dynamical system $\left(\mathrm{S},\ \mathrm{K},\ \boldsymbol{\phi}_{\mathbf{p}}\right)$ is the change of the symbolic TQ-image of the dynamical system that satisfies the condition:

$$\Gamma_a^{\mathrm{TQ}} \xrightarrow[\mathbf{p}=\mathbf{p}_b]{\text{TQ-bif}} \Gamma_b^{\mathrm{TQ}}, \quad \Gamma_a^{\mathrm{TQ}} \neq \Gamma_a^{\mathrm{TQ}}, \quad \mathbf{p}_a \neq \mathbf{p}_b.$$

Where Γ_a^{TQ} and Γ_b^{TQ} are symbolic TQ-images of the dynamical system $\left(\mathrm{S},\ \mathrm{K},\ \boldsymbol{\phi}_{\mathbf{p}}\right)$ before and after bifurcation respectively, and $\mathbf{p}_b$ is the bifurcation value of the vector of parameters.

Let us introduce the bifurcation kernel $\Gamma_{ab}^{\mathrm{TQ}}$:

$$\Gamma_{ab}^{\mathrm{TQ}} = \Gamma_a^{\mathrm{TQ}} \wedge \Gamma_b^{\mathrm{TQ}} : \ \Gamma_a^{\mathrm{TQ}} \backslash \Gamma_{ab}^{\mathrm{TQ}} \cap \Gamma_b^{\mathrm{TQ}} \backslash \Gamma_{ab}^{\mathrm{TQ}} = \varnothing, \quad \Gamma_{ab}^{\mathrm{TQ}} = \left\langle \mathrm{V}_{ab}^{\Gamma},\ \mathrm{E}_{ab}^{\Gamma} \right\rangle.$$

In addition to the graph $\Gamma_{ab}^{\mathrm{TQ}}$, define the set of arches in the graphs Γ_a^{TQ} and Γ_b^{TQ} respectively, that are, at their beginnings and ends, incident to the vertices of the graph $\Gamma_{ab}^{\mathrm{TQ}}$:

$${}^{a}\mathrm{E}_{ab}^{\Gamma} = \left\{ (v',\ v'') \in \mathrm{E}_a^{\Gamma} : \ v' \in \mathrm{V}_{ab}^{\Gamma},\ v'' \in \mathrm{V}_{ab}^{\Gamma} \right\},$$

$${}^{b}\mathrm{E}_{ab}^{\Gamma} = \left\{ (v',\ v'') \in \mathrm{E}_b^{\Gamma} : \ v' \in \mathrm{V}_{ab}^{\Gamma},\ v'' \in \mathrm{V}_{ab}^{\Gamma} \right\}.$$

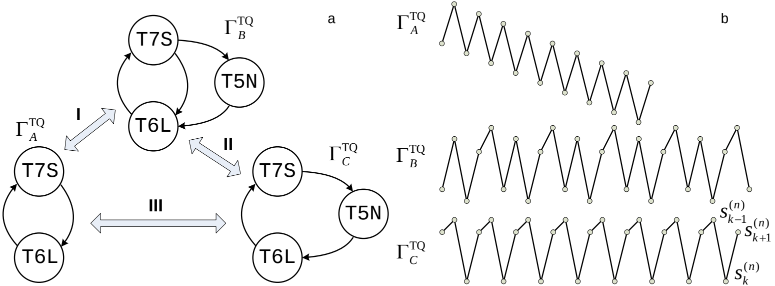
The classification of the TQ-bifurcations

Thus, the TQ-bifurcations can be divided into three genres:

$$I \qquad \mathrm{V}_a^\Gamma \neq \mathrm{V}_b^\Gamma, \quad {}^a\mathrm{E}_{ab}^\Gamma = {}^b\mathrm{E}_{ab}^\Gamma;$$

$$II \qquad \mathrm{V}_a^\Gamma = \mathrm{V}_b^\Gamma, \quad {}^a\mathrm{E}_{ab}^\Gamma \neq {}^b\mathrm{E}_{ab}^\Gamma;$$

$$III \qquad \mathrm{V}_a^\Gamma \neq \mathrm{V}_b^\Gamma, \quad {}^a\mathrm{E}_{ab}^\Gamma \neq {}^b\mathrm{E}_{ab}^\Gamma.$$

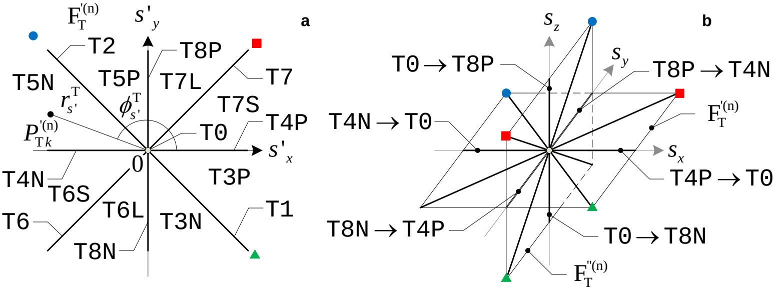


The nature of the TQ-bifurcations suggest that they define homogenous dynamics areas of the system $(\mathrm{S}, \mathrm{K}, \boldsymbol{\phi}_\mathbf{p})$ from the standpoint of the symbolic CTQ-analysis, i.e., with regard to the trajectory shape of the dynamical system in the extended space of states $\mathrm{S} \times \mathrm{K}$.

The detection of the TQ-bifurcations

Let us introduce the designation $P^{(n)}_{\mathrm{Q}\,\mathbf{s}} = \left(\mathbf{s}_x^{(n)},\, \mathbf{s}_y^{(n)},\, \mathbf{s}_z^{(n)}\right)$:

$$\begin{aligned} \mathbf{s}_x &= \mathbf{s} - \mathbf{f}(\mathbf{s},\, \mathbf{p}), \\ \mathbf{s}_y &= \mathbf{f}^2(\mathbf{s},\, \mathbf{p}) - \mathbf{f}(\mathbf{s},\, \mathbf{p}), \\ \mathbf{s}_z &= \mathbf{f}^3(\mathbf{s},\, \mathbf{p}) - \mathbf{f}^2(\mathbf{s},\, \mathbf{p}). \end{aligned} \tag{3}$$



The detection of the TQ-bifurcations

Thus, the T-symbols and transitions between T-symbols can be detected elementarily. For example:

$$\texttt{T5P}: \quad s_x < 0,\ s_y > 0,\ -s_x < s_y,$$

$$\texttt{T5P} \to \texttt{T6L}: \quad s_x < 0,\ s_y > 0,\ s_z < 0,\ -s_x < s_y,\ -s_z < s_y.$$

It is also possible the simultaneous identification of period-cycles 1–3 in map $\mathbf{s}_{k+1} = \mathbf{f}\big(\mathbf{s}_k,\, \mathbf{p}\big)$. Conditions are represented as follows:

$$1: \quad s_x = s_y = s_z = 0,$$

$$2: \quad s_x = s_y = -s_z \neq 0,$$

$$3: \quad -s_x + s_y + s_z = 0,\ s_x \neq 0,\ s_y \neq 0,\ s_z \neq 0.$$

Note that, in light of the form of similar-terms representation in formulas, such an analysis is best performed out in modern computer algebra systems. This saves a lot of time and eliminates many common mistakes.

Outline section

The basic properties

$$s_{k+1} = 4\,\lambda\, s_k(1 - s_k), \quad s \in (0,\, 1), \quad \lambda \in (0,\, 1],$$

$$\lambda > \lambda_\infty = 0.892486418\ldots \text{ – chaos zone}$$

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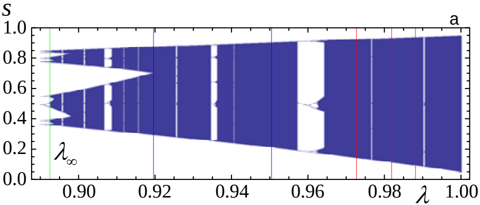
The Bifurcation Diagram:



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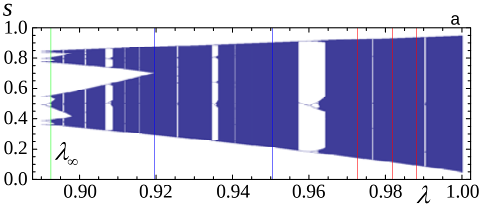
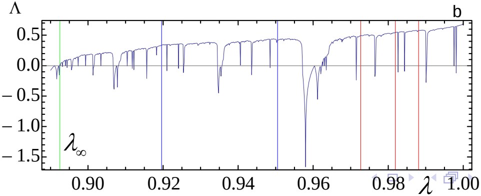
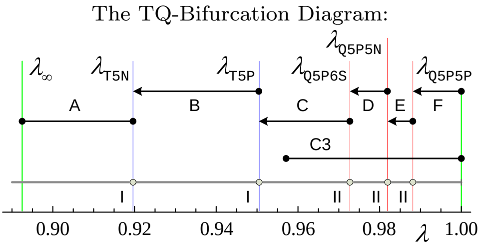


The Largest Lyapunov Exponent:



The TQ-bifurcations



The TQ-bifurcations

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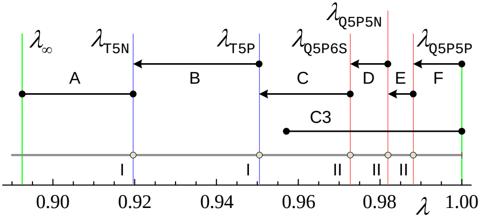
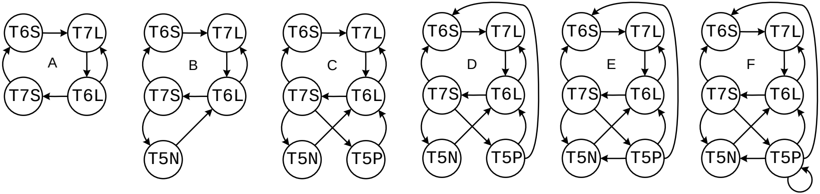


The TQ-Images:



The exact values of bifurcation parameters are given in the full paper.

Outline section

Summary

- Identification of related TQ-bifurcations and period-cycles 1–3 in map $\mathbf{s}_{k+1} = \mathbf{f}\big(\mathbf{s}_k,\, \mathbf{p}\big)$ by means of its representation in the coordinates of the representative point $P^{(n)}_{\mathrm{Q}\,\mathbf{s}}$ allows, with regard to the propositions of the Sharkovsky theorem, to carry out comprehensive analysis of the system in order to determine various properties of chaos, including the systems synchronization, system controllability, and the complexity of its trajectory $\{\mathbf{s}_k\}$.
- A constructive application of the proposed approach to the analysis of discrete dynamical systems is potentially possible in the field of research issues, such as: transmission and encoding of information in chaotic systems, controlling the chaotic dynamic and suppressing chaotic oscillations by means of small external perturbations, analyzing properties of various discrete lattices.

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Thank you for your attention!