Estimation complexity of chaotic oscillations in aspect of the shape of their trajectories

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Outline

Outline section

Why is Complexity?

Complexity:

– is one of the fundamental scientific concepts.

– is important of information-structural characteristic any object.

Is no exception and a more narrow notion of the:

"complexity of dynamic process".

Complexity, Science and Society, eds. Bogg J., Geyer R., Radcliffe Publishing, 2007.

Measurements of Complexity, Eds. L. Petiti, A. Vulpiari, Lect. Notes in Phys., 1988, 314.

Computation of complexity – it is an open question...

However, along with this, questions of definition and calculation the complexity of dynamic processes remains of methodologically open:

- 1877 Year – Ludwig Boltzmann introduced the notion of "entropy"
- R. Hartley and C. Shannon – gave entropy of sense information
- A.N. Kolmogorov and Y.G. Sinai – entropy generalized to the dynamical systems
- Nonlinear Dynamics – Lyapunov exponents, Kolmogorov entropy, S-parameter Klymontovich
- A.N. Kolmogorov – an algorithmic approach to the concept of "complexity"
- Radio physics – a time-frequency criterion of complexity
- V.I. Arnold – calculation complexity of latticed sequences of the form $\mathbb{Z}_2 \times \mathbb{Z}$
- ...

Main problems of approaches (as a rule):

- some do not allow you to measure complexity of a particular trajectory;
- some are very laborious to compute and interpret the results;
- some energy is measured and not the information;
- some do not carry over on $\mathbb{R}^N \times \mathbb{Z}$-the continual process.

Yet another approach...

Denote the discrete dynamical system:

$$\mathbf{s}_{k+1} = \mathbf{f}\left(\mathbf{s}_k,\, \mathbf{p}\right), \quad \{\mathbf{s}_k\}_{k=1}^{K},$$

$$\mathbf{s} \in \mathrm{S} \subset \mathbb{R}^N, \quad k \in \mathrm{K} \subset \mathbb{N}, \quad \mathbf{p} \in \mathrm{P} \subset \mathbb{R}^M, \quad n = \overline{1,\, N},\, k = \overline{1,\, K},\, m = \overline{1,\, M}.$$

Key Idea

More complex dynamic process, has a more complex shape of the trajectory in space $\mathrm{S} \times \mathrm{K}$. (ideologically close of perimetric complexity)

Perimetric complexity is a measure of the complexity of binary pictures. The concept of perimetric complexity was first introduced by:

F. Attneave and M.D. Arnoult, *The Quantitative Study of Shape and Pattern Perception* // Psychological Bulletin, 53 (6), 1956, pp. 452-471. http://psycnet.apa.org/journals/bul/53/6/452.

Detection and analysis shape of the trajectory in space $\mathrm{S} \times \mathrm{K}$ of sequence $\{\mathbf{s}_k\}_{k=1}^{K}$ – via symbolic CTQ-analysis.

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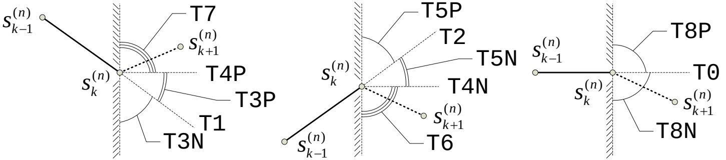
Definition of alphabet

We introduce the primary mapping:

$$\left\{\mathbf{s}_{k-1}^{(n)},\, \mathbf{s}_{k}^{(n)},\, \mathbf{s}_{k+1}^{(n)}\right\} \Rightarrow T_k^{\alpha\varphi}|_n, \quad T_k^{\alpha\varphi} = [T_k^{\alpha\varphi}|_1 \ldots T_k^{\alpha\varphi}|_N], \quad \{T_k^{\alpha\varphi}\}_{k=1}^{K},$$

where $T^{\alpha\varphi}|_n$ – symbol of T-alphabet:

$$\mathrm{T}_o^{\alpha\varphi} = \{\mathtt{T0}, \mathtt{T1}, \mathtt{T2}, \mathtt{T3N}, \mathtt{T3P}, \mathtt{T4N}, \mathtt{T4P}, \mathtt{T5N}, \mathtt{T5P}, \mathtt{T6}, \mathtt{T7}, \mathtt{T8N}, \mathtt{T8P}\}.$$



Additionally define the $Q^{\alpha\varphi}|_n$ – symbol of Q-alphabet ($\mathrm{Q}_o^{\alpha\varphi} \ni Q^{\alpha\varphi}|_n$):

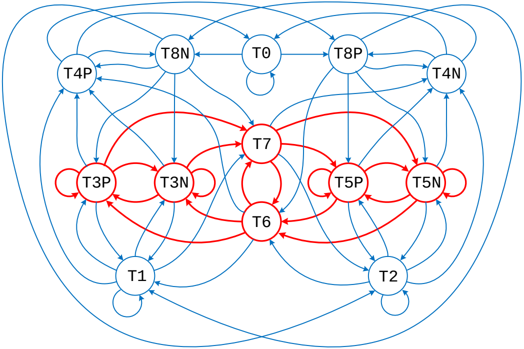
$$Q_k^{\alpha\varphi}|_n \equiv T_k^{\alpha\varphi}|_n \to T_{k+1}^{\alpha\varphi}|_n, \quad Q_k^{\alpha\varphi} = [Q_k^{\alpha\varphi}|_1 \ldots Q_k^{\alpha\varphi}|_N], \quad \{Q_k^{\alpha\varphi}\}_{k=1}^{K}.$$

Symbolic TQ-image

Symbolic TQ-image of sequence $\{\mathbf{s}_k\}_{k=1}^{K}$

Directed graph $\Gamma^{TQ}|_n = \langle \mathrm{V}^{\Gamma}|_n,\, \mathrm{E}^{\Gamma}|_n \rangle$:

$\mathrm{V}^{\Gamma}|_n \subseteq \mathrm{T}_o^{\alpha\varphi}$ – vertex $\Gamma^{TQ}|_n$ and $\mathrm{E}^{\Gamma}|_n \subseteq \mathrm{Q}_o^{\alpha\varphi}$ – edges $\Gamma^{TQ}|_n$.



The primary formalisms:

- CTQ-symmetry;
- TQ-bifurcations;
- TQ-complexity;
- T-synchronization;
- Q-prediction;
- Q-control.

Main Article and Talk (in English)

- A.V.M., *Structure of Synchronized Chaos Studied by Symbolic Analysis in Velocity–Curvature Space* // Technical Physics Letters, 38:2 (2012), 155—159, arXiv: 1203.4214.
- A.V.M., *Multidimensional Dynamic Processes Studied by Symbolic Analysis in Velocity–Curvature Space* // Computational Mathematics and Mathematical Physics, 52:7 (2012), 1017—1028.
- A.V.M., *Measure of Synchronism of Multidimensional Chaotic Sequences Based on Their Symbolic Representation in a T-Alphabet* // Technical Physics Letters, 38:9 (2012), 804–808, arXiv: 1212.2724.

- A.V.M., *The possibilities by symbolic analysis in velocity-curvature space: TQ-bifurcation, symmetry, synchronization* // School-Seminar "Interaction of Mathematics and Physics: New Perspectives"/ Proceedings. – Moscow, Steklov Mathematical Institute, 2012.

Outline section

1. Motivation
2. The symbolic CTQ-analysis
   - Main Constructions
   - Additional information
3. TQ-complexity
   - The general idea of approach
   - The measures of complexity
4. The study oscillator Rossler
   - Description of the problem
   - Results of the experiment
5. Conclusion

Specific complexity of the symbols

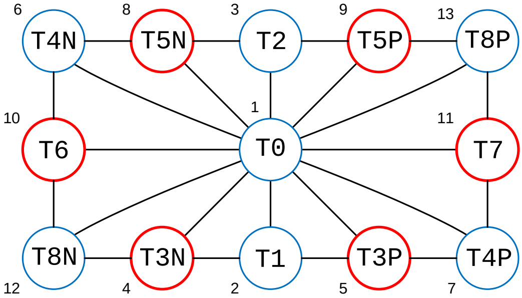
The specific complexity of the symbol $T^{\alpha\varphi}|_n$:

$T^{\alpha\varphi}\|_n$	T0	T1, T2	T4∗, T8∗	T3∗, T5∗	T6, T7
$C^T\|_n$	1	2	4	5	6

The specific complexity of the symbol $Q^{\alpha\varphi}|_n$:

$$C^Q|_n = \mathrm{d_T}\left(T_k^{\alpha\varphi}|_n,\, T_{k+1}^{\alpha\varphi}|_n\right) + 1,$$

where $\mathrm{d_T}(\cdot,\cdot)$ – the shortest path between two vertices in a graph $\mathrm{D_T}$:



Degenerate measures of complexity

Topological $\mathbf{C}^{dt}_{\Gamma}|_n = \left[C^{dt}_{\Gamma T}|_n,\, C^{dt}_{\Gamma Q}|_n\right]^{\mathrm{T}}$:

$$C^{dt}_{\Gamma\circ}|_n = \sum_{*} \operatorname{sign} \Delta^*|_n;$$

Metrical $\mathbf{C}^{dm}_{\Gamma}|_n = \left[C^{dm}_{\Gamma T}|_n,\, C^{dm}_{\Gamma Q}|_n\right]^{\mathrm{T}}$:

$$C^{dm}_{\Gamma\circ}|_n = \exp H^{\Gamma\circ}|_n;$$

where:

$$H^{\Gamma\circ}|_n = -\sum_{*} \Delta^*|_n \ln \Delta^*|_n,$$

$$* \in \mathrm{T}^{\alpha\varphi}_{o} : \circ = T, \quad * \in \mathrm{Q}^{\alpha\varphi}_{o} : \circ = Q,$$

$H^{\Gamma\circ}|_n$ – Boltzmann-Shannon entropy
of $\mathrm{V}^{\Gamma}|_n$ and $\mathrm{E}^{\Gamma}|_n$ components of graph $\Gamma^{TQ}|_n$.

Weighted measures of complexity

<u>Topological</u> $\mathbf{C}_{\Gamma}^{wt}|_n = \left[C_{\Gamma T}^{wt}|_n,\, C_{\Gamma Q}^{wt}|_n\right]^{\mathrm{T}}$:

$$C_{\Gamma\circ}^{wt}|_n = \sum_{*} C^*|_n \operatorname{sign} \Delta^*|_n;$$

<u>Metrical</u> $\mathbf{C}_{\Gamma}^{wm}|_n = \left[C_{\Gamma T}^{wm}|_n,\, C_{\Gamma Q}^{dm}|_n\right]^{\mathrm{T}}$:

$$C_{\Gamma\circ}^{wm}|_n = \begin{cases} 0 & C_{\Gamma\circ}^{wt}|_n = 0, \\ \exp \tilde{H}^{\Gamma\circ}|_n & \text{otherwise}. \end{cases};$$

where:

$$\tilde{H}^{\Gamma\circ}|_n = \frac{1}{1-q} \ln \sum_{*} C^*|_n \left(\tilde{\Delta}^*|_n\right)^q, \quad \tilde{\Delta}^*|_n = \frac{\hat{\Delta}^*|_n}{\sum_{*} C^*|_n\, \hat{\Delta}^*|_n}, \quad \hat{\Delta}^*|_n = \frac{\left(\Delta^*|_n\right)^{b_\circ}}{C^*|_n},$$

$$b_\circ = \frac{\ln C_{\Gamma\circ}^{wt}|_n - \ln C^*|_n}{\ln C_{\Gamma\circ}^{dt}|_n}, \quad q \propto 1 + \ln \frac{\max C^*|_n}{\min C^*|_n},$$

$$* \in \mathrm{T}_o^{\alpha\varphi} : \circ = T, \quad * \in \mathrm{Q}_o^{\alpha\varphi} : \circ = Q,$$

$\tilde{H}^{\Gamma\circ}|_n$ – weighted Renyi entropy of graph $\Gamma^{TQ}|_n$.

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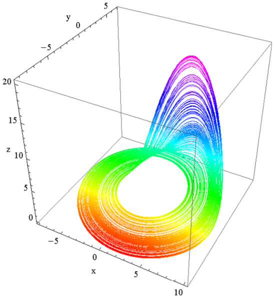
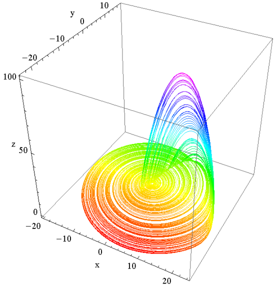
Problem statement

Rossler system:

$$\dot{x} = -y - z, \quad \dot{y} = x + p\,y, \quad \dot{z} = q + z(x - r), \quad p = 0.2, \quad q = 0.1,$$

band-type chaos: $r = r_b = 4.4$ screw-type chaos: $r = r_s = 12$





STC more complex than BTC

R. Gilmore, M. Lefranc, The topology of chaos. Wiley-Interscience, 2002.
O.E. Rossler // Bulletin of Mathematical Biology, 1977, 39(2), 275–289.

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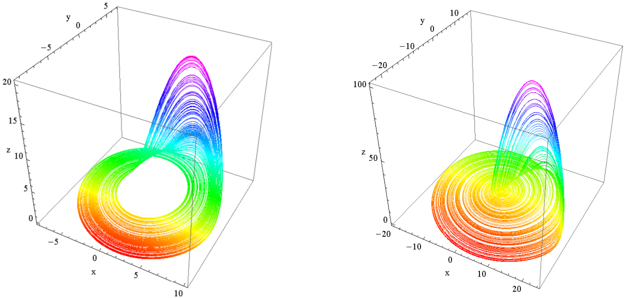

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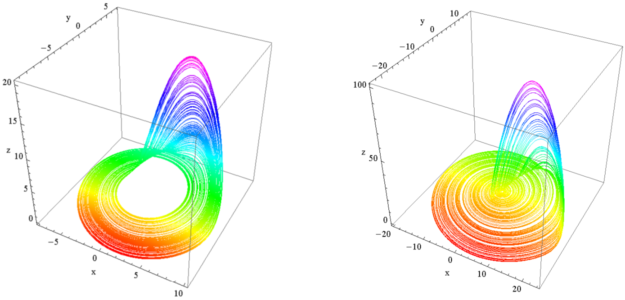


STC – contains a Shilnikov homoclinic orbits

L.P. Shilnikov, A. Shilnikov, D. Turaev and L. Chua, Methods of Qualitative Theory in Nonlinear Dynamics. Part I, II. World Sci. 1998, 2001.

Description of the numerical experiment

Method: Explicit Runge-Kutta, 5 order, Fixed Step.

Period: $T = [0,\, 8 \times 10^3]$.

Step: $\Delta t = 10^{-2}$.

Quantity trajectories: $N = 1000$ (for each mode: BTC and STC).

Initial conditions: $x_0 = \xi_1 \in [-7,\, 7]$, $y_0 = \xi_2 \in [-7,\, 7]$, $z_0 = \xi_3 \in [0,\, 15]$.

ξ_{1-3} – uncorrelated random variable with uniform distribution.

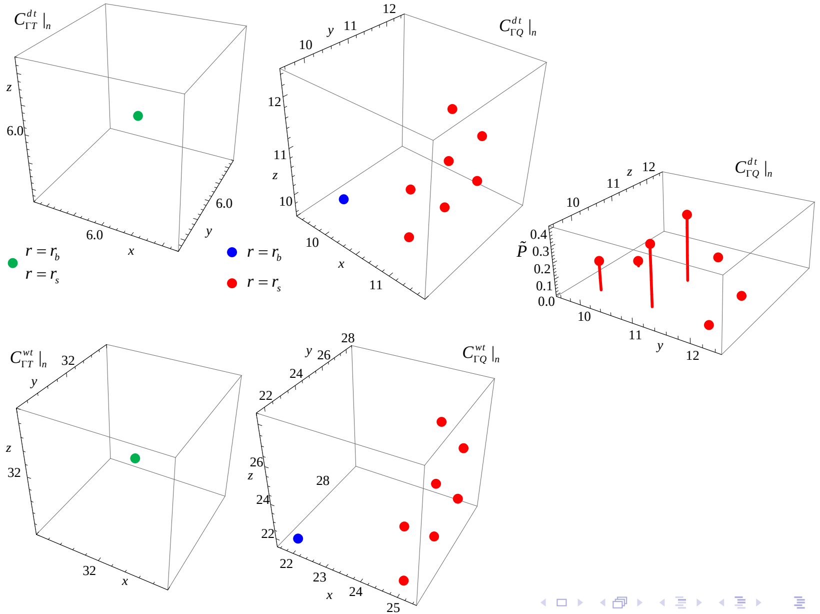
Analytical Period: $T' = [7,\, 8] \times 10^3$.

Quantity counts in sequence $\{T_k^{\alpha\varphi}|_n\}_{k=1}^{K}$: $K = 10^5$.

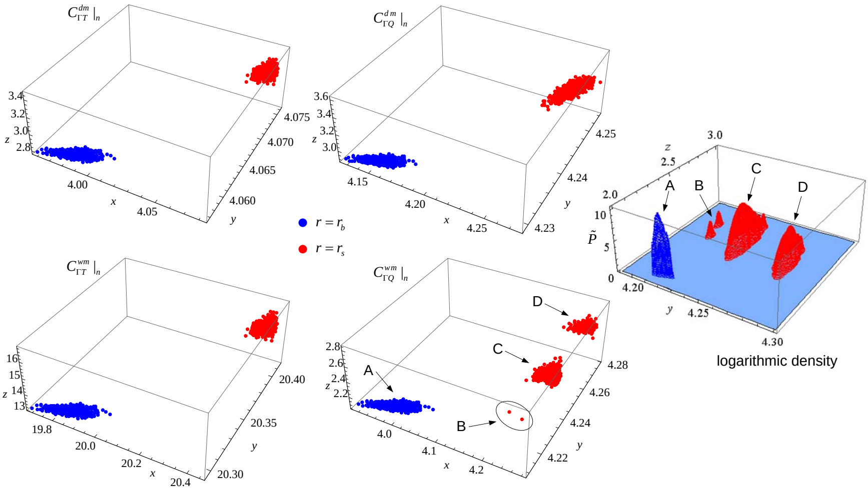
The boundaries of intervals of random variables (by probability): $\beta = 0.999$.

All calculations and visualization conducted at Wolfram Mathematica 9.

Topological Measures

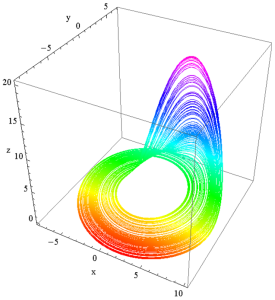


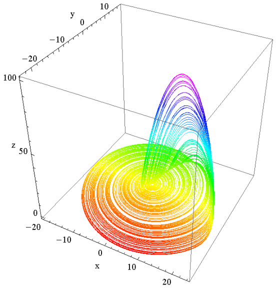
Metrical Measures



Short Discussion

band-type chaos: $r = r_b = 4.4$ screw-type chaos: $r = r_s = 12$





STC more complex than BTC

Open question:

Which trajectories belong to clusters B, C, D, and why do they occur?

Outline section

Summary

- In this report, we proposed an original approach to the evaluation and analysis the complexity of chaotic sequences.
- The numerical experiment demonstrated the efficiency measures of TQ-complexity.
- The developed tools allows methods of computational physics us to study various phenomena in nonlinear multi-dimensional dynamical systems.
- At the moment, we decide to two open questions:
  - Binding of the free parameter q (in weighted Renyi entropy of graph $\Gamma^{TQ}|_n$).
  - Formation of a measure of TQ-complexity for scalar estimating multi-dimensional systems.

Thank you for your attention!