A Constructive Subdivision of TQ-space

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Outline

Outline section

Definition of the T-alphabet

Denote the discrete dynamical system:

$$\mathbf{s}_{k+1} = \mathbf{f}\left(\mathbf{s}_k,\, \mathbf{p}\right), \quad \boldsymbol{\phi}_{\mathbf{p}} : \mathrm{S} \times \mathrm{K} \to \mathrm{S}, \quad \boldsymbol{\phi}_{\mathbf{p}}(\mathbf{s},\, k) \equiv \mathbf{f}^k\left(\mathbf{s},\, \mathbf{p}\right), \quad \left\{\mathbf{s}_k\right\}_{k\in\mathrm{K}},$$

$$\mathbf{s} \in \mathrm{S} \subset \mathbb{R}^N, \quad k \in \mathrm{K} \subseteq \mathbb{Z}, \quad \mathbf{p} \in \mathrm{P} \subset \mathbb{R}^L, \quad \mathbf{f} \in \mathrm{C}^0(\mathrm{S} \times \mathrm{P}), \quad n = \overline{1,\, N},\, l = \overline{1,\, L}.$$

We introduce the main map:

$$\left\{\mathbf{s}_{k-1}^{(n)},\, \mathbf{s}_k^{(n)},\, \mathbf{s}_{k+1}^{(n)}\right\} \Rightarrow T_k^{\alpha\varphi}|_n, \quad T_k^{\alpha\varphi} = \left[T_k^{\alpha\varphi}|_1 \ldots T_k^{\alpha\varphi}|_N\right], \quad \left\{T_k^{\alpha\varphi}\right\}_{k\in\mathrm{K}},$$

where $T^{\alpha\varphi}$ – symbol of T-alphabet:

$$T^{\alpha\varphi} \in \mathrm{T}_o^{\alpha\varphi} = \mathrm{T}_0^{\alpha\varphi} \cup \mathrm{T}_s^{\alpha\varphi} \cup \mathrm{T}_c^{\alpha\varphi}.$$

It is unambiguously divided into three separate classes:

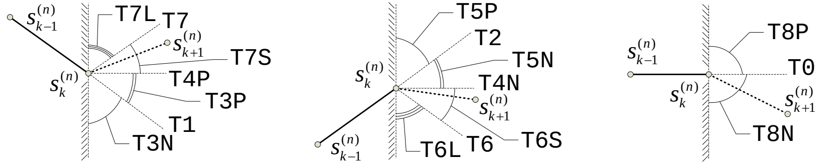
$$\mathrm{T}_0^{\alpha\varphi} = \{\mathtt{T0}\},$$

$$\mathrm{T}_s^{\alpha\varphi} = \{\mathtt{T1},\, \mathtt{T2},\, \mathtt{T4N},\, \mathtt{T4P},\, \mathtt{T6},\, \mathtt{T7},\, \mathtt{T8N},\, \mathtt{T8P}\},$$

$$\mathrm{T}_c^{\alpha\varphi} = \{\mathtt{T3N},\, \mathtt{T3P},\, \mathtt{T5N},\, \mathtt{T5P},\, \mathtt{T6S},\, \mathtt{T6L},\, \mathtt{T7S},\, \mathtt{T7L}\}.$$

Definition of the T-alphabet

Geometry of the T-alphabet symbols:



Definition of the Q-alphabet

Table of the admissible transitions $T_k^{\alpha\varphi}|_n \to T_{k+1}^{\alpha\varphi}|_n \equiv Q_k^{\alpha\varphi}|_n$:

$k+1$ / k	T0	T1	T2	T3N	T3P	T4N	T4P	T5N	T5P	T6S	T6	T6L	T7S	T7	T7L	T8N	T8P
T0																	
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T2																	
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The TQ-Image of discrete dynamical system

Definition 1

The directed graph $\Gamma^{\mathrm{TQ}} = \langle \mathrm{V}^{\Gamma},\, \mathrm{E}^{\Gamma} \rangle$, implies that:

$$\mathrm{V}^{\Gamma} \subseteq \mathrm{T}_o^{\alpha\varphi}|N, \quad \mathrm{E}^{\Gamma} \subseteq \mathrm{Q}_o^{\alpha\varphi}|N,$$

$$\forall\, T^{\alpha\varphi} \in \mathrm{V}^{\Gamma} \Rightarrow \exists\, \mathbf{s}_0 \in \mathrm{S} : T_1^{\alpha\varphi} = T^{\alpha\varphi},\ \forall\, \mathbf{s}_0 \in \mathrm{S} \Rightarrow \exists\, T^{\alpha\varphi} \in \mathrm{V}^{\Gamma} : T_1^{\alpha\varphi} = T^{\alpha\varphi},$$

$$\forall\, Q^{\alpha\varphi} \in \mathrm{E}^{\Gamma} \Rightarrow \exists\, \mathbf{s}_0 \in \mathrm{S} : Q_1^{\alpha\varphi} = Q^{\alpha\varphi}, \forall\, \mathbf{s}_0 \in \mathrm{S} \Rightarrow \exists\, Q^{\alpha\varphi} \in \mathrm{E}^{\Gamma} : Q_1^{\alpha\varphi} = Q^{\alpha\varphi},$$

is a *Complete symbolic TQ-image* of the dynamical system $\left(\mathrm{S},\, \mathrm{K},\, \boldsymbol{\phi}_{\mathbf{p}}\right)$.

Let us also define the adjacency matrix $\mathbf{A}^{\mathrm{TQ}}$ of the graph vertices Γ^{TQ}.

Then matrix $\mathbf{A}^{\mathrm{TQ}}$ may be interpreted as a matrix of valid transitions between states of a topological Markov chain.

Thus, the sequence $\{T_k^{\alpha\varphi}\}_{k\in\mathrm{K}}$ may be considered from the standpoint of topological Markov chain defined by matrix $\mathbf{A}^{\mathrm{TQ}}$.

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But there is problem! The T-alphabet is too rough (uninformative) for study sequence $\{T_k^{\alpha\varphi}\}_{k\in\mathrm{K}}$ as topological Markov chain.

Main Article and Talk (in English)

- A.V. Makarenko, "Analysis of phase synchronization of chaotic oscillations in terms of symbolic CTQ-analysis", *Technical Physics*, vol. 61, no. 2, pp. 265–273, 2016.
- A.V. Makarenko, "Analysis of the time structure of synchronization in multidimensional chaotic systems", *Journal of Experimental and Theoretical Physics*, vol. 120, no. 5, pp. 912–921, 2015.
- A.V. Makarenko, "Generalized synchronization of multidimensional chaotic systems in terms of symbolic CTQ-analysis", *The Proceedings of 8th Chaotic Modeling and Simulation International Conference*, Paris, ISAST, IHP, 2015, pp. 477–490.
- A.V. Makarenko, "TQ-Bifurcations in Discrete Dynamical Systems: Analysis of Qualitative Rearrangements of the Oscillation Mode", *Journal of Experimental and Theoretical Physics*, 2016 [in press].
- A.V. Makarenko, "The TQ-bifurcation in Discrete Dynamical Systems. General Properties", *Proceedings of International Conference Stability and Oscillations of Nonlinear Control Systems*, ICS RAS, Moscow, 2016.
- A.V. Makarenko, "Estimation of the TQ-complexity of chaotic sequences", *IFAC-PapersOnLine*, vol. 48, no. 11, pp. 1049–1055, 2015.

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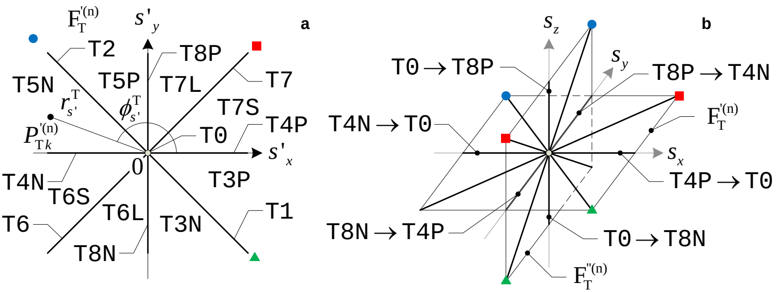
Definition

Let us introduce the designation $P^{(n)}_{\mathrm{Q}\,\mathbf{s}} = \left(\mathbf{s}^{(n)}_x,\, \mathbf{s}^{(n)}_y,\, \mathbf{s}^{(n)}_z\right) \in \mathbb{R}^3$:

$$\mathbf{s}_x = \mathbf{s} - \mathbf{f}(\mathbf{s},\, \mathbf{p}),$$
$$\mathbf{s}_y = \mathbf{f}^2(\mathbf{s},\, \mathbf{p}) - \mathbf{f}(\mathbf{s},\, \mathbf{p}),$$
$$\mathbf{s}_z = \mathbf{f}^3(\mathbf{s},\, \mathbf{p}) - \mathbf{f}^2(\mathbf{s},\, \mathbf{p}).$$



Let us also define the full TQ-space:

$$\mathrm{P}_{\mathrm{TQ}} = \mathrm{P}^{(1)}_{\mathrm{TQ}} \times \cdots \times \mathrm{P}^{(n)}_{\mathrm{TQ}} \times \cdots \times \mathrm{P}^{(N)}_{\mathrm{TQ}}, \quad \mathrm{P}^{(n)}_{\mathrm{TQ}} = \Pr_n \mathrm{P}_{\mathrm{TQ}}, \quad \dim \mathrm{P}_{\mathrm{TQ}} = 3\,N.$$

Basic application

Thus, the T-symbols and Q-symbols can be detected elementarily.
For example:

$$\texttt{T5P}: \quad s_x < 0,\ s_y > 0,\ -s_x < s_y,$$

$$\texttt{T5P} \to \texttt{T6L}: \quad s_x < 0,\ s_y > 0,\ s_z < 0,\ -s_x < s_y,\ -s_z < s_y.$$

It is also possible the simultaneous identification of period-cycles 1–3 in map $\mathbf{s}_{k+1} = \mathbf{f}\big(\mathbf{s}_k,\ \mathbf{p}\big)$. Conditions are represented as follows:

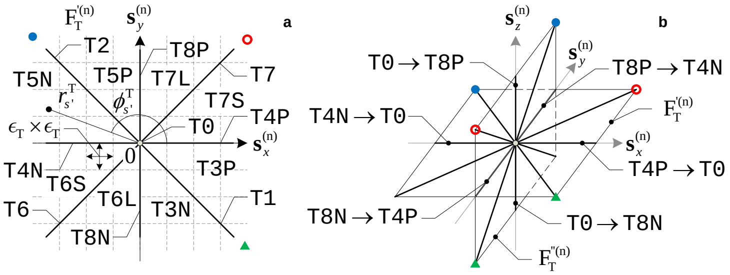
$$1: \quad s_x = s_y = s_z = 0,$$

$$2: \quad s_x = s_y = -s_z \neq 0,$$

$$3: \quad -s_x + s_y + s_z = 0,\ s_x \neq 0,\ s_y \neq 0,\ s_z \neq 0.$$

Note that, in light of the form of similar-terms representation in formulas, such an analysis is best performed out in modern computer algebra systems. This saves a lot of time and eliminates many common mistakes.

Partition principle



The subdivision of the subspace $\mathrm{P}^{(n)}_{\mathrm{TQ}}$ into disjoint cubes:

$$\mathrm{S}^{(n)}_x|m_x \times \mathrm{S}^{(n)}_y|m_y \times \mathrm{S}^{(n)}_z|m_z, \quad m_\circ \in \mathbb{Z}, \quad \epsilon_\mathrm{T} > 0, \qquad \mathrm{S}^{(n)}_\circ|m_\circ = \epsilon_\mathrm{T} \begin{cases} (m_\circ - 1,\, m_\circ] & m_\circ > 0, \\ 0 & m_\circ = 0, \\ [m_\circ,\, m_\circ + 1) & m_\circ < 0. \end{cases},$$

determines natural extension over the T- and Q-alphabets, with codes of each alphabet symbol extended with the indices $m_\circ$, that set the number (and effectively the location) of the cell.

Note that this subdivision preserves the Hausdorff dimension of the symbols in the T- and Q-alphabets in the subspace $\mathrm{P}^{(n)}_{\mathrm{TQ}}$.

Main realization

The consecutive subdivision of the space P_{TQ} through refinement of the cells

$$\mathrm{S}_x^{(n)}|m_x \times \mathrm{S}_y^{(n)}|m_y \times \mathrm{S}_z^{(n)}|m_z$$

due to the parameter $\epsilon_{\mathrm{T}} \to 0$ results in a sequence of the graphs $\left\{\Gamma_{\epsilon_{\mathrm{T}}}^{\mathrm{TQ}}\right\}_{\epsilon_{\mathrm{T}} \in \mathcal{E}}$.

One of the simplest subdivision schemes is the dichotomous scheme.

$$\epsilon_{\mathrm{T}\,0}^{(n)} = \max\left|\left(\left\{\mathbf{s}_x^{(n)},\, \mathbf{s}_y^{(n)},\, \mathbf{s}_z^{(n)}\right\} : \, \mathbf{s} \in \mathrm{S}_{\mathrm{a}}\right)\right|,$$

$$\epsilon_{\mathrm{T}\,r+1}^{(n)} = \epsilon_{\mathrm{T}\,r}^{(n)}/2, \quad m_\circ = \overline{-d,\, 0,\, d}, \quad d = 2^r,$$

where S_{a} is the analyzed range of definition of the phase variable $\mathbf{s}$, for example, the attractor of the system $\left(\mathrm{S},\, \mathrm{K},\, \boldsymbol{\phi}_{\mathbf{p}}\right)$.

When certain requirements are observed with regard to the nature of ϵ_{T} change and the encoding scheme for $m_\circ$ indices, the subdivision of the space P_{TQ} can be constructively applied to studying various properties of dynamical systems.

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Summary

In conclusion, it is worth mentioning that the proposed subdivision of the space P_{TQ} provides an opportunity to significantly increase the depth of performing certain tasks in terms of the symbolic CTQ-analysis:

- Studying the T-synchronization of chaotic systems;
- Estimating TQ-complexity of the trajectories of dynamical systems;
- Analyzing the TQ-bifurcations in discrete systems;
- etc.

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Thank you for your attention!